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A Product Decomposition of the Fundamental Solution of a Second Order Parabolic Equation

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1. INTRODUCTION

The purpose of this paper is to establish a decomposition of the fundamental solution of a second order parabolic equation of the form

$$\frac{\partial}{\partial t} f(x, t) = L_{x,t} f(x, t), \tag{1}$$

where $x \in R^n$ (Euclidean n-dimensional space), $t \in (0, \infty)$, $f: R^n \times (0, \infty) \to R$, and $L_{x,t}$ is a second order elliptic operator on R^n of the form

$$L_{x,t} = L_{x,t}^p + L_{x,t}^q . \tag{2}$$

We assume that $p + q = n$, and so, if we consider $R^n = R^p \oplus R^q$, then for $x \in R^n$ we have a unique decomposition $x = x^p + x^q$ with $x^p \in R^p$ and $x^q \in R^q$. $L_{x,t}^p$ is a second order elliptic operator based in R^p and $L_{x,t}^q$ a second order elliptic operator based in R^q. Specifically, if we let $P : R^n \to R^p$ be the projection on R^p and $Q : R^n \to R^q$ be the projection on R^q, then we require that $L_{x,t}^p$ be of the form

$$\begin{aligned} L_{x,t}^p f(x, t) = {} & \mathrm{Tr}_{R^n}[[PA^p(x, t)\, P][D^2 f(x, t)]] \\ & + [PB^p(x, t)] \cdot [Df(x, t)] + C^p(x, t) f(x, t). \end{aligned} \tag{3}$$

D denotes the Fréchet derivative of f with respect to x, and $\cdot$ denotes the dot product in R^n. We further require that for all $(x, t) \in R^n \times (0, \infty)$,

1.A $A^p(x, t)$ is a symmetric member of $L(R^n, R^n)$, there exists a constant $\lambda > 0$ (independent of x and t) such that $A^p(x, t) \geqslant \lambda I$ (here I is the identity in $L(R^n, R^n)$), and $A^p(x, t) = A^p(x^p, t)$;

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1.B $B^p(x, t) \in R^n$ and $B^p(x, t) = B^p(x^p, t)$;

1.C $C^p(x, t) \in R$ and $C^p(x, t) = C^p(x^p, t)$.

We impose analogous requirements on $L^q_{x,t}$ (simply replace p and P by q and Q in the requirements for $L^p_{x,t}$).

For $i = p$ or q, $L^i_{x,t}$ determines (by restriction) an elliptic operator $S^i_{x,t}$ in R^i. For example, when $f : R^p \times (0, \infty) \to R$, we have

$$S^p_{x,t}f(x, t) = \mathrm{Tr}_{R^p}[[PA^p(x, t)\, P][D^2f(x, t)]] + [PB^p(x, t)] \cdot [Df(x, t)] + C^p(x, t)\, f(x, t),$$

whenever the right side exists. Here we note that $PA^p(x, t)P : R^p \to R^p$ and so may be regarded as a member of $L(R^p, R^p)$, and similarly $PB^p(x, t) \in R^p$. D now refers to the Fréchet derivative of f with respect to x, where x varies over R^p. Thus $D^2f(x, t) \in L(R^p, R^p)$ and $Df(x, t) \in R^p$. Letting $Z^i(x, t; y, s)$ denote the fundamental solution of $\partial/\partial t\, f(x, t) = S^i_{x,t}f(x, t)$, and $Z(x, t; y, s)$ that of $\partial/\partial t\, f(x, t) = L_{x,t}f(x, t)$, we will show that

$$Z(x, t; y, s) = Z^p(x^p, t; y^p, s)Z^q(x^q, t; y^q, s)$$

subject to the usual regularity hypotheses on the coefficients of $L_{x,t}$.

Example. If $L_{x,t}f(x, t) = \mathrm{Tr}_{R^n}[AD^2f(x, t)]$, where A is a positive definite operator on R^n, then the fundamental solution of $\partial/\partial t f(x, t) = L_{x,t}f(x, t)$ is given by

$$Z(x, t; y, s) = \frac{[\det A]^{-\frac{1}{2}}}{[4\pi(t - s)]^{n/2}} \exp[-(A^{-1}(x - y), x - y)/4(t - s)].$$

If R^p and R^q are orthogonal reducing subspaces for A, such that $p + q = n$, then letting A^i denote the restriction of A to R^i ($i = p, q$) we observe that $Z = Z^pZ^q$, where

$$Z^i(x, t; y, s) = \frac{[\det A^i]^{-\frac{1}{2}}}{[4\pi(t - s)]^{i/2}} \exp[-([A^i]^{-1}(x^i - y^i), x^i - y^i)/4(t - s)].$$

The fundamental solution of $\partial/\partial t f = L_{x,t}f$ is often written in the form

$$Z(x, t; y, s) = W(x, t; y, s) + \int_0^t \int_{R^n} W(x, t; z, r)F(z, r; y, s)\, dz\, dr,$$

where both W and F are real-valued functions defined on $\{0 \leqslant s < t < \infty,$ $x \in R^n,\ y \in R^n\}$. W is referred to as the parametrix and F as the perturbing kernel (see, e.g. Refs. [1, 2, and 5]). W and F are by no means unique—the W and F of Ref. [1] are different from those of Ref. [5]. Usually W is similar to the fundamental solution of the differential equation under consideration where the second-order coefficients are fixed at a particular point in $R^n \times (0, \infty)$ and the first and zero order terms are set equal to zero. In a situation such as we are considering, where the coefficients are direct-sum decomposable, it is obvious from the explicit form usually given for W (very similar to the Z of the above example) that W is decomposable as a product. It is not obvious whether a similar decomposition is valid for the perturbing term.

Our interest in obtaining a product decomposition is for the purpose of developing approximation techniques for the study of second-order parabolic equations on an infinite-dimensional real separable Hilbert space (considered as one logical generalization of R^n). It is then hoped that, by studying certain properties of such equations on suitable (e.g. Riemannian-Wiener) infinite-dimensional manifolds, it will eventually be possible to obtain Hodge's theorem for such manifolds.

The type of approximation technique which we have in mind is as follows. Let H be a real separable Hilbert space with inner product $(\, , \,)$. We consider a second order parabolic equation of the form $\partial/\partial t f(x, t) = L_{x,t} f(x, t)$, where $L_{x,t} f(x, t) = \operatorname{Tr}_H[A(x, t) D^2 f(x, t)] + (B(x, t), Df(x, t)) + C(x, t) f(x, t)$. It is easy to formulate appropriate ellipticity hypotheses on $L_{x,t}$. At the present time the existence of a fundamental solution for this equation has been established only for very special cases (* in particular, the coefficients are temporally homogeneous, B and C are zero, and $A(x) - I$ is of the trace class). Most finite-dimensional techniques for studying properties of the fundamental solution, and thereby establishing regularity properties for the solutions of the differential equation, will not extend to the infinite-dimensional case. In particular, finite-dimensional techniques for establishing nonnegativity and the semigroup property of the fundamental solution are inappropriate. One would, therefore, like to argue for the validity of these properties from their finite-dimensional validity. In Ref. [4], this is accomplished for the case (*) mentioned above in the presence of several smoothness hypotheses on the coefficients. Basically, $L_{x,t}$ is approximated by a differential operator $L^K_{x,t}$ acting in a finite-dimensional subspace K of H plus the Laplacian Δ^K acting in $K^\perp$. If P is the orthogonal projection of H onto K, $L^K_{x,t} f \equiv \operatorname{Tr}_K[PA(Px)PD^2 f]$. The fundamental solution of $\partial/\partial t f = [L^K_{x,t} \oplus \Delta^K] f$ is represented as the product of the fundamental solutions of $\partial/\partial t\, f = L^K_{x,t} f$ and $\partial/\partial t f = \Delta^K f$. A great deal is known about the properties of the fundamental solution of $\partial/\partial t f = \Delta^K f$, and also if we replace Δ^K by an elliptic operator with constant coefficients. Letting $P \to I$ (the identity operator on H) in an appropriate fashion, it can be shown that the fundamental solution of $\partial/\partial t f = [L^K_{x,t} \oplus \Delta^K] f$ converges to that of $\partial/\partial t f = L_{x,t} f$ in total variation norm (in infinite dimensions we must consider fundamental solutions as measures on H rather than, as is the case in finite dimensions, the Radon–Nikodym derivatives of such measures with respect to Lebesgue measure). It is then immediate from the product decomposition of the fundamental solution of the "semifinite" approximation that the non-negativity and semigroup properties of the individual components of the approximating fundamental solution imply the validity of those properties for the fundamental solution of $\partial/\partial t f = L_{x,t} f$.

It is believed that the introduction of first and zero order terms and of time-dependence is a straightforward generalization of the present existence proof for the fundamental solution of $\partial/\partial t f = L_{x,t} f$ (the major anticipated difficulty is notational complication). It was not readily apparant that the approximation technique of Ref. [4] would generalize to such cases; the existence of a product decomposition for the approximating fundamental solutions being basic to the technique. This paper demonstrates that the product decomposition is not peculiar to the special case considered in Ref. [4] and indicates the feasibility of "semifinite" approximation techniques in studying considerably more general infinite-dimensional parabolic equations.

2. A Uniqueness Theorem

The fundamental solution of a parabolic equation is usually obtained as the sum of an infinite series of functions (see, e.g., [1, 2, and 5]) and, when it is so formulated, the property which we desire is not at all evident except in the special case where one of the decomposed portions has constant coefficients. Instead, then, of studying the form of the fundamental solution, we will appeal to a uniqueness theorem of Il'in, Kalashnikov and Oleinik [3] to establish our decomposition. The required theorem is as follows:

THEOREM. *Consider the differential equation*

$$\operatorname{Tr}_{R^n}[A(x, t)\, D^2 f(x, t)] + B(x, t) \cdot Df(x, t)$$

$$+ C(x, t) f(x, t) - \frac{\partial}{\partial t} f(x, t) = 0, \tag{4}$$

where (x, t) varies over $H = \{(x, t) : x \in R^n, 0 < t \leqslant T\}$, $A(x, t)$ is a symmetric member of $L(R^n, R^n)$, $B(x, t) \in R^n$ and $C(x, t) \in R$. If

2.A *there exists a constant* $\gamma > 0$ *such that* $A(x, t) \geqslant \gamma I$ *for all* $(x, t) \in \bar{H}$ ($\equiv$ *the closure of* H),

2.B $A(x, t)$, $B(x, t)$ *and* $C(x, t)$ *are bounded and continuous on* $\bar{H}$,

2.C *for each* $0 \leqslant t \leqslant T$, $A(x, t)$, $B(x, t)$ *and* $C(x, t)$ *satisfy a Hölder condition with respect to* x *on* R^n,

2.D *for each* $x \in R^n$, $A(x, t)$ *satisfies a Hölder condition with respect to* t *on* $0 < t \leqslant T$,

then there exists a unique real-valued function $Z(x, t; y, s)$ *possessing the following properties*:

3.A Z, $\partial Z/\partial t$ *and* $\partial^2 Z/\partial x_i\, \partial x_j$ $(i, j = 1, 2, ..., n)$ *are continuous in* $\{0 \leqslant s < t \leqslant T, x \in R^n, y \in R^n\}$,

3.B *for each* s *and* y, Z *satisfies Eq.* (4) *with respect to the variables* x *and* t,

3.C *for each* $\epsilon > 0$, Z *is bounded on the set* $\{t - s + | x - y | \geqslant \epsilon\}$,

3.D *for any continuous bounded function* f *on* R^n *and for each* $s \in [0, T)$, *we have*

$$\lim_{t \downarrow s} \int_{R^n} Z(x, t; y, s) f(y)\, dy = f(x),$$

the convergence being uniform on compact subsets of R^n.

Under conditions 2.A–2.D, Z *also satisfies*

4.A $Z(x, t; y, s)$ *is everywhere positive for* $0 \leqslant s < t \leqslant T$, *and*

4.B *there exist positive constants* C *and* c *such that* $Z(x, t; y, s) \leqslant C(t - s)^{-n/2} \exp[-| x - y |^2/[c(t - s)]]$ *for all* $0 \leqslant s < t \leqslant T$, x *and* $y \in R^n$.

3. The Decomposition Theorem

THEOREM. *Let the differential operator* $L_{x,t}$ *be of the form in Eq.* (2), *with* $L^p_{x,t}$ *and* $L^q_{x,t}$ *satisfying Eq.* (3) *and conditions* 1.A–1.C. *Setting*

$$A(x, t) \equiv PA^p(x, t)P + QA^q(x, t)Q,$$

$$B(x, t) \equiv PB^p(x, t) + QB^q(x, t),$$

and

$$C(x, t) \equiv C^p(x, t) + C^q(x, t),$$

assume that A, B *and* C *satisfy* 2.A–2.D. *If* $S^i_{x,t}$ *is the restriction of* $L^i_{x,t}$ *to* R^i $(i = p, q)$, *and if* $Z^i(x, t; y, s)$ *denotes the fundamental solution of* $\partial/\partial t f(x, t) = S^i_{x,t} f(x, t)$ *in* R^i, *then the fundamental solution* $Z(x, t; y, s)$ *of Eq.* (1) *satisfies*

$$Z(x, t; y, s) = Z^p(x^p, t; y^p, s) Z^q(x^q, t; y^q, s). \tag{5}$$

Proof. We need only check that $Z^p Z^q$ satisfies the properties 3.A–3.D in each finite t-interval $(0, T]$. Uniqueness will then imply that Eq. (5) is satisfied for all $t \in (0, \infty)$.

The verification of property 3.A is immediate. Property 3.C follows trivially from the observation that

$$\begin{aligned}
&\{t - s + | x - y | \geqslant \epsilon\} \\
&\quad \subseteq \{t - s \geqslant \epsilon/2\} \cup \{| x^p - y^p |^2 + | x^q - y^q |^2 \geqslant \epsilon^2/4\} \\
&\quad \subseteq \{t - s \geqslant \epsilon/2\} \cup \{| x^p - y^p |^2 \geqslant \epsilon^2/8\} \cup \{| x^q - y^q |^2 \geqslant \epsilon^2/8\} \\
&\quad = \{t - s \geqslant \epsilon/2\} \cup \{| x^p - y^p | \geqslant \epsilon/2\sqrt{2}\} \cup \{| x^q - y^q | \geqslant \epsilon/2\sqrt{2}\} \\
&\quad \subseteq \{t - s + | x^p - y^p | \geqslant \epsilon/2\sqrt{2}\} \cup \{t - s + | x^q - y^q | \geqslant \epsilon/2\sqrt{2}\}
\end{aligned}$$

and the fact that Z^p and Z^q each satisfy property 3.C.

To show property 3.B, write $x \in R^n$ as $(x_1, ..., x_n)$ where $x^p = (x_1, ..., x_p, 0, ..., 0)$ and $x^q = (0, ..., 0, x_{p+1}, ..., x_n)$. Let $W(x, t; y, s)$ be defined by the right side of Eq. (5). $DW(x, t; y, s)$ may be represented by an $n \times 1$ matrix relative to the standard basis for R^n and $D^2W(x, t; y, s)$ as an $n \times n$ matrix. We have

$$\begin{aligned}
DW(x, t; y, s) &= \left(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, ..., \frac{\partial W}{\partial x_n}\right)\Big/ (x, t; y, s), \\
&= \left(\frac{\partial Z^p}{\partial x_1}, ..., \frac{\partial Z^p}{\partial x_p}, 0, ..., 0\right)\Big/_{(x^p, t; y^p, s)} Z^q(x^q, t; y^q, s) \\
&\qquad + Z^p(x^p, t; y^p, s) \left(0, ..., \frac{\partial Z^q}{\partial x_{p+1}}, ..., \frac{\partial Z^q}{\partial x_n}\right)\Big/_{(x^q, t; y^q, s)}
\end{aligned}$$

and

$$[PA^p(x, t)\, P][D^2W(x, t; y, s)] = \begin{bmatrix} A_{11}^p & \cdots & A_{1p}^p & \\ \vdots & & \vdots & 0 \\ A_{p1}^p & \cdots & A_{pp}^p & \\ & 0 & & 0 \end{bmatrix} \begin{bmatrix} & & \\ & D^2W & \\ & & \end{bmatrix},$$

$$= \begin{bmatrix} A_{11}^p & \cdots & A_{1p}^p & \\ \vdots & & \vdots & 0 \\ A_{p1}^p & \cdots & A_{pp}^p & \\ & 0 & & 0 \end{bmatrix} \left[\begin{array}{c:c} D^2Z^p & 0 \\ \hdashline 0 & 0 \end{array}\right] Z^q,$$

$$= \left[\begin{array}{c:c} [PA^pP][D^2Z^p] & 0 \\ \hdashline 0 & 0 \end{array}\right] Z^q,$$

where all terms are to be evaluated at $(x, t; y, s)$. D^2Z^p and $[PA^pP][D^2Z^p]$ are considered as $p \times p$ matrices. Similarly,

$$[QA^q(x, t)\, Q][D^2W(x, t; y, s)] = Z^p \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & [QA^qQ][D^2Z^q] \end{array}\right].$$

Thus

$$L_{x,t}^p W = \{\mathrm{Tr}_{R^p}[PA^pP][D^2Z^p] + [PB^p] \cdot [DZ^p] + C^pZ^p\}\, Z^q,$$

$$= (S_{x,t}^p Z^p)\, Z^q,$$

and, similarly,

$$L_{x,t}^q W = Z^p(S_{x,t}^q Z^q).$$

It now follows that

$$L_{x,t}W - \frac{\partial W}{\partial t} = \left(S_{x,t}^p Z^p - \frac{\partial}{\partial t} Z^p\right) Z^q + Z^p \left(S_{x,t}^q Z^q - \frac{\partial}{\partial t} Z^q\right),$$

$$= 0.$$

To obtain property 3.D we will proceed by cases.

Case 1. For each $y \in R^n$, $f(y) = f_p(y^p) f_q(y^q)$, where f_i is bounded and continuous on R^i $(i = p, q)$. Then

$$\begin{aligned}
&\lim_{t \downarrow s} \int_{R^n} W(x, t; y, s) f(y)\, dy \\
&\quad = \left\{ \lim_{t \downarrow s} \int_{R^p} Z^p(x^p, t; y^p, s) f_p(y^p)\, dy^p \right\} \cdot \left\{ \lim_{t \downarrow s} \int_{R^q} Z^q(x^q, t; y^q, s) f_q(y^q) dy^q \right\} \\
&\quad = f_p(x^p) f_q(y^q) \\
&\quad = f(x). \qquad (6)
\end{aligned}$$

If S is a compact subset of R^n, then S is contained in a set of the form $S_p \times S_q$, where S_i is a compact subset of R^i. Since

$$\lim_{t \downarrow s} \int_{R^i} Z^i(x^i, t; y^i, s) f_i(y^i)\, dy^i = f_i(x^i)$$

uniformly on S_i (Z^i satisfies property 3.D) it follows that the convergence of the left side of Eq. (6) is uniform on S.

Case 2. f is continuous on R^n and vanishes outside a compact set S of the form $S_p \times S_q$, with S_i a closed bounded ball in R^i. On S we can apply the Stone–Weierstrass theorem to approximate f uniformly by functions f^k $(k = 1, 2,...)$ which are finite linear combinations of functions of the form treated in case 1, i.e., $f^k(y) = \sum_{j=1}^m f_p^{k,j}(y^p) f_q^{k,j}(y^q)$. We may extend $f_i^{k,j}$ from S_i to all of R^i in a continuous bounded fashion as follows. If S_i has center a_i and radius r_i, and if $y \in R^i/S_i$, then define

$$f_i^{k,j}(y) = f_i^{k,j} \left(r_i \frac{y - a_i}{|y - a_i|} \right).$$

We now have

$$\begin{aligned}
&\left| \int_{R^n} W(x, t; y, s) f(y)\, dy - f(x) \right| \\
&\qquad \leqslant \left| \int_{R^n} W(x, t; y, s)(f(y) - f^k(y))\, dy \right| \\
&\qquad\quad + \left| \int_{R^n} W(x, t; y, s) f^k(y)\, dy - f^k(x) \right| + |f^k(x) - f(x)| \\
&\qquad = (i) + (ii) + (iii), \quad \text{say.}
\end{aligned}$$

Given $\epsilon > 0$, we may choose k_ϵ so that $(iii) < \epsilon$ for $k = k_\epsilon$ and for all $x \in S$.

For $k = k_\epsilon$, there exists a $\delta > 0$ such that $t - s \leqslant \delta$ implies $(ii) < \epsilon$, for all $x \in S$ (by Case 1). In order to estimate (i), we note that $\| f - f^k \|_\infty = \sup_{y \in S} | f(y) - f^k(y)|$, since $f(y)$ vanishes outside S and for y outside S $f^k(y)$ has the same value as f^k evaluated at that boundary point of S which is closest to y. Thus, for $k = k_\epsilon$,

$$(i) \leqslant \epsilon \int_{R^n} W(x, t; y, s)\, dy.$$

Each Z^i satisfies condition 4.B, and without loss of generality we may assume that the constants C and c are the same for both Z^p and Z^q. An elementary calculation then gives

$$\int_{R^n} W(x, t; y, s)\, dy \leqslant C(\pi c)^{n/2} \int_{R^n} [\pi c(t-s)]^{-n/2} \exp[-\,|\, x - y\,|^2/[c(t-s)]]\, dy$$
$$= C(\pi c)^{n/2}.$$

Noting that this calculation is independent of x, t and s, it follows that W satisfies property 3.D.

Case 3. f is continuous and bounded on R^n. To show that property 3.D holds uniformly for x in a compact set S, we may without loss of generality assume that S is of the form considered in Case 2. If S^i has center a_i and radius r_i, let U_i be the closed ball in R^i which center a_i and radius $r_i + 1$, and V_i be the closed ball in R^i with center a_i and radius $r_i + 2$. Let g_i be a continuous function defined on R^i satisfying

$$\begin{aligned} g_i(y) &= 1 \qquad \text{for} \quad y \in U_i \\ 0 \leqslant g_i(y) &\leqslant 1 \qquad \text{for all} \quad y \in R^i \\ g_i(y) &= 0 \qquad \text{for} \quad y \notin V_i \,. \end{aligned}$$

Set $g(y) = g_p(y^p) g_q(y^q)$ and $U = U_p \times U_q$. We can now write $f = gf + (1 - g)f$. Then, for $x \in S$,

$$\left| \int_{R^n} W(x, t; y, s) f(y)\, dy - f(x) \right| \leqslant \left| \int_{R^n} W(x, t; y, s)\, g(y) f(y)\, dy - g(x) f(x) \right|$$
$$+ \left| \int_{R^n/U} W(x, t; y, s)(1 - g(y)) f(y)\, dy \right|$$
$$= (i) + (ii), \text{ say}.$$

$(i) \to 0$ as $t \downarrow s$, by Case 2, the convergence being uniform for $x \in S$. Also

$$(ii) \leqslant \|f\|_\infty \int_{R^n/U} W(x, t; y, s)\, dy.$$

If, for any Borel set B in R^n, we define

$$\mu_t(B) = \int_B (2\pi t)^{-n/2} \exp[- \mid y \mid^2/2t]\, dy,$$

then it is well-known that for each fixed $\delta > 0$, $\mu_t(\mid y \mid \geqslant \delta) = o(t)$ as $t \downarrow 0$. Thus, for all $x \in S$, we have

$$\int_{R^n/U} W(x, t; y, s)\, dy \leqslant C(\pi c)^{n/2} \int_{|y| \geqslant 1} [\pi c(t-s)]^{-n/2} \exp[- \mid y \mid^2/[c(t-s)]]\, dy$$
$$= o(t - s) \quad \text{as} \quad t \downarrow s,$$

and it follows that W satisfies property 3.D.

Thus $W = Z$, the fundamental solution of Eq. (1).

Remark. Although the decompositions of $A(x, t)$ into $PA^p(x, t)P + QA^q(x, t)Q$ and $B(x, t)$ into $PB^p(x, t) + QB^q(x, t)$ must be unique (for a given P), that of $C(x, t)$ into $C^p(x, t) + C^q(x, t)$ is not unique. If, e.g., $C(x, t) = C$ (a constant), then $C^p(x, t)$ and $C^q(x, t)$ could be any two constants which add up to C.

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